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# Conformal O(3,2) symmetry of the two-dimensional inverse square potential

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**Abstract.** It is shown that the dynamical systems describing a point mass moving in a repulsive inverse square potential in a plane and a free relativistic massless particle are isomorphic to each other. The obvious conformal invariance of the massless particle appears as a hidden dynamical symmetry of the inverse square potential.

## 1. Introduction

It is known in the literature [1-8] that the mechanical problem of a non-relativistic point mass moving in an inverse square potential possesses a scalar O(2, 1) invariance algebra. This is spanned by the conserved quantities

$$\mathcal{H}=\frac{1}{2\mu}|\boldsymbol{P}|^2+\frac{\beta}{|\boldsymbol{R}|^2} \qquad \mathcal{D}=\tfrac{1}{2}(R_kP^k-2t\mathcal{H})$$
$$\mathcal{K}=-t^2\mathcal{H}-2t\mathcal{D}+\tfrac{1}{2}\mu|\boldsymbol{R}|^2 \qquad (|\boldsymbol{R}|>0) \tag{1}$$

which satisfy the Poisson bracket relations:

$$\{\mathcal{H},\mathcal{D}\}=-\mathcal{H} \qquad \{\mathcal{D},\mathcal{K}\}=-\mathcal{K} \qquad \{\mathcal{H},\mathcal{K}\}=-2\mathcal{D}. \tag{2}$$

Using this 'dynamical symmetry algebra', together with the obvious rotational invariance, one can give a group theoretical derivation of important quantities in the quantum mechanical version of the inverse square potential problem [2, 3]. This sort of situation is familiar from the study of the Coulomb problem (e.g. [9] and references therein) for which, in $n$-dimensional space, the complete dynamical group is $O(n+1,2)$. The hidden symmetry of the Coulomb problem can be made explicit [10-13] by converting it into that of a free particle moving on a sphere or a hyperboloid, depending on the sign of the energy, in $(n+1)$-dimensional space. In analogy (and rather amusingly), here I show that a non-relativistic point mass moving in a plane under the influence of a repulsive inverse square potential can be transformed into a free relativistic massless particle by a canonical transformation which also preserves the respective energies and angular momenta. The obvious conformal invariance of the massless particle, which amounts to an O(3, 2) algebra in the two-dimensional case, appears as a hidden dynamical symmetry of the inverse square potential. The generators of relativistic time translation, dilatation and timelike 'special conformal transformation' span a scalar O(2,1) subalgebra of the conformal O(3,2) which is essentially identical to the O(2,1) given by equations (1) and (2) above.

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In § 2 we shall go through the conformal algebra of the free massless particle. Then in § 3 I shall exhibit the equivalence with the inverse square potential problem.

## 2. The conformal invariance of a free massless scalar particle

Let $T^*Q$ be the cotangent bundle of the three-dimensional Minkowski space $Q$. It carries the standard symplectic form

$$\omega = -\mathrm{d}\theta \qquad \theta = p_\mu \, \mathrm{d}x^\mu \qquad (\mu = 0, 1, 2) \tag{3}$$

where $x^\mu = (x^0, \boldsymbol{r})$, $p^\mu = (p^0, \boldsymbol{p})$ are coordinates with respect to a fixed inertial frame in which $g_{\mu\nu} = \mathrm{diag}(-1,1,1)$. The 'evolution space' (throughout the paper I follow the terminology of Souriau [14]) $\mathscr{E}_0^+$ of a free spinless particle of mass 0 is a hypersurface in $T^*Q$ defined by the constraint

$$m^2 = -g_{\mu\nu}p^\mu p^\nu = 0 \qquad p^0 > 0. \tag{4}$$

The motions of the particle give rise to a fibration of $\mathscr{E}_0^+$. A particular motion can be specified by giving the vector $\boldsymbol{r}$ at which the corresponding worldline meets the $x^0 = 0$ hyperplane and $\boldsymbol{p}$, the spacelike part of its conserved momentum. The 'Lagrange form' [14] $\omega_{|\mathscr{E}_0^+}$ descends to a symplectic form on the 'space of motions' $O_0^+ = \mathbb{R}^2 \times (\mathbb{R}^2 \backslash \{0\})$ which is in our coordinates $\mathrm{d}r^i \wedge \mathrm{d}p_i$. The infinitesimal generators of the conformal group of $(Q, g_{\mu\nu})$ are

$$\begin{aligned} T_\mu &= \partial/\partial x^\mu & M_{\mu\nu} &= x_\mu \, \partial/\partial x^\nu - x_\nu \, \partial/\partial x^\mu \\ D &= x^\alpha \, \partial/\partial x^\alpha & K_\mu &= 2x_\mu x^\alpha \, \partial/\partial x^\alpha - x^\alpha x_\alpha \, \partial/\partial x^\mu . \end{aligned} \tag{5}$$

These, when lifted to $T^*Q$, preserve $\theta$ as well as the constraint (4). Thus they descend to Hamiltonian vector fields on the symplectic manifold $(O_0^+, \mathrm{d}r^i \wedge \mathrm{d}p_i)$. The corresponding Hamiltonians (components of the moment map of the action of O(3, 2) on $O_0^+$) are given explicitly in the parameters $(\boldsymbol{r}, \boldsymbol{p})$ as follows:

$$\begin{aligned} &\mathscr{T}_0 = T_0 \lrcorner\, \theta = -|\boldsymbol{p}| \qquad \mathscr{T}_i = T_i \lrcorner\, \theta = p_i \qquad (i = 1, 2) \\ &\mathscr{M}_{12} = M_{12} \lrcorner\, \theta = \varepsilon_{ij} r^i p^j \qquad \mathscr{M}_{0i} = M_{0i} \lrcorner\, \theta = |\boldsymbol{p}| r^i \\ &\mathscr{D} = D \lrcorner\, \theta = r_i p^i \qquad \mathscr{K}_0 = K_0 \lrcorner\, \theta = |\boldsymbol{p}| \cdot |\boldsymbol{r}|^2 \\ &\mathscr{K}_i = K_i \lrcorner\, \theta = (2r_k p^k) r_i - |\boldsymbol{r}|^2 p_i . \end{aligned} \tag{6}$$

## 3. Equivalence through convenient coordinates

Here we consider a non-relativistic point particle of mass $\mu$ which moves in a plane and is acted upon by a repulsive force deriving from an inverse square potential. Let the triplet $(\boldsymbol{R}, \boldsymbol{P}, t)$ stand for the general element of the evolution space $\mathscr{E} = (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}^2 \times \mathbb{R}$ of the particle. Its energy $\mathscr{H}$ is given by (1) with $\beta > 0$. All the information about the mechanical problem considered here is encoded [14] in the Lagrange form $\Omega$ defined on $\mathscr{E}$ by the formula

$$\Omega = \mathrm{d}R^i \wedge \mathrm{d}P_i + \mathrm{d}\mathscr{H} \wedge \mathrm{d}t. \tag{7}$$

An arbitrarily given $(\boldsymbol{R}, \boldsymbol{P}, t)$ determines, as an initial value, a scattering trajectory of the particle. So on every trajectory there is a unique turning point. Let $\boldsymbol{F}(\boldsymbol{R}, \boldsymbol{P}, t)$ and

$\tau(\boldsymbol{R}, \boldsymbol{P}, t)$ be, respectively, the unit vector pointing to the turning point and the time when it is reached by the particle. A straightforward calculation results in the explicit formulae

$$\begin{aligned}\tau &= t - R^k P_k / 2\mathscr{H} \\ F_i &= (R_i \cos\alpha + \varepsilon_{ij} R^j \sin\alpha)/|\boldsymbol{R}|\end{aligned} \tag{8a}$$

where

$$\alpha = \frac{L}{(2\mu\beta + L^2)^{1/2}} \tan^{-1}\left(\frac{R^k P_k}{(2\mu\beta + L^2)^{1/2}}\right) \qquad L = \varepsilon_{ij} R^i P^j. \tag{8b}$$

Now let $\mathscr{N}$ denote the space of motions (the quotient of $\mathscr{E}$ by the classical motions) of the problem under investigation. The conserved quantities $\tau$, $\boldsymbol{F}$, $L$, $\mathscr{H}$ provide us with a smooth parametrisation of $\mathscr{N}$. The Poisson brackets of these observables are

$$\begin{aligned}&\{\mathscr{H}, L\} = \{\mathscr{H}, F_i\} = \{F_1, F_2\} = \{\tau, F_i\} = 0 \\ &\{\mathscr{H}, \tau\} = 1 \qquad \{L, F_i\} = \varepsilon_{ij} F^j\end{aligned} \tag{9}$$

as is easy to check from (8). Now let us introduce new 'convenient' coordinates on $\mathscr{N}$:

$$\xi_i = -\tau F_i + L\varepsilon_{ij} F^j / \mathscr{H} \qquad \zeta_i = \mathscr{H} F_i. \tag{10}$$

By virtue of (9) and (10), the Poisson brackets of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are of the standard form

$$\{\xi_i, \zeta_j\} = \delta_{ij} \qquad \{\xi_i, \xi_j\} = \{\zeta_i, \zeta_j\} = 0. \tag{11}$$

Thus we can define a canonical transformation between $O_0^+$ and $\mathscr{N}$ by the equation

$$\boldsymbol{r} = \boldsymbol{\xi} \qquad \boldsymbol{p} = \boldsymbol{\zeta}. \tag{12}$$

Then, using this map, we can convert the conformal algebra of the free massless particle into a symmetry algebra of the inverse square potential problem. In terms of the variables ($L$, $\mathscr{H}$, $\tau$, $F_i$) the generators of this 'dynamical symmetry algebra' take the following form:

$$\begin{aligned}&\mathscr{T}_0 = -\mathscr{H} \qquad \mathscr{M}_{12} = L \qquad \mathscr{D} = -\tau\mathscr{H} \qquad \mathscr{K}_0 = (L^2 + \mathscr{D}^2)/\mathscr{H} \\ &\mathscr{T}_i = \mathscr{H} F_i \qquad M_{0i} = \mathscr{D} F_i + L\varepsilon_{ij} F^j \\ &\mathscr{K}_i = 2\mathscr{D} L \varepsilon_{ij} F^j / \mathscr{H} + (\mathscr{D}^2 - L^2) F_i / \mathscr{H}.\end{aligned} \tag{13}$$

The interesting point is that, as can be seen from (1), (8) and (13), in addition to being a canonical transformation our map (12) also carries the respective energies, angular momenta and dilatation generators into each other. The generators $\mathscr{T}_0 = -\mathscr{H}$, $\mathscr{D}$, $\mathscr{K}_0$ span an O(2,1) subalgebra of the conformal O(3,2). This is 'essentially identical' to the O(2,1) algebra given by (1) and (2). To see this, first let us observe that the transformation

$$\mathscr{K} \rightarrow \mathscr{K} + f(L)/\mathscr{H} \tag{14}$$

leaves the Poisson bracket relations at (2) unchanged for any smooth function $f$ of $L$. The point is that the generator of relativistic 'timelike special conformal transformations' $\mathscr{K}_0$ is related to its non-relativistic analogue $\mathscr{K}$ by a transformation of this kind. In fact, the following relation:

$$\mathscr{K}_0 = \mathscr{K} + (\tfrac{3}{4} L^2 - \tfrac{1}{2}\mu\beta)/\mathscr{H} \tag{15}$$

holds, as is easily verified. So our dynamical O(3,2) symmetry algebra can be thought of as an extension of the O(2,1) invariance algebra of the inverse square potential which was known previously [1-8]. One has the O(2,1) invariance in any dimensions; the existence of the extension given here seems to be a rather peculiar property of the two-dimensional case. In this case, one could construct a single unitary irreducible representation of O(3,2) out of all the scattering states of a point mass moving in a repulsive inverse square potential background. For example, one could use the methods of geometric quantisation [14, 15] to construct the representation in question out of the homogeneous symplectic manifold $(\mathcal{N}, \mathrm{d}\xi_i \wedge \mathrm{d}\zeta^i) \simeq (O_0^+, \mathrm{d}r_i \wedge \mathrm{d}p^i)$.

It is easy to extend the map (12) to a one-to-one map between the evolution spaces $\mathscr{E}_0^+$ and $\mathscr{E}$ which carries the respective Lagrange forms $\omega_{|\mathscr{E}_0^+}$ and $\Omega$ into each other. To achieve this one simply has to identify, in addition to (12), the respective time coordinates $x^0$ and $t$.

In conclusion, we have shown that the classical mechanics of a non-relativistic massive test particle moving in a plane in a repulsive inverse square potential is equivalent to that of a free relativistic massless particle.

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## References

[1] Niederer U 1974 *Helv. Phys. Acta* **47** 167
[2] de Alfaro V, Fubini S and Furlan G 1976 *Nuovo Cimento* A **34** 569
[3] Jackiw R 1980 *Ann. Phys., NY* **129** 183
[4] Duval C 1982 *These de doctorat d'Etat* Marseille
[5] Horváthy P A 1983 *Lett. Math. Phys.* **7** 353
[6] Hussin V and Sinzinkayo J 1985 *J. Math. Phys.* **26** 1072
[7] Hussin V and Jacques M 1986 *J. Phys. A: Math. Gen.* **19** 3471
[8] D'Hoker E and Vinet L 1985 *Commun. Math. Phys.* **97** 391
[9] Englefield M 1972 *Group Theory and the Coulomb Problem* (New York: Wiley)
[10] Gyorgyi G 1968 *Nuovo Cimento* A **53** 717
[11] Moser J 1970 *Commun. Pure Appl. Math.* **23** 609
[12] Souriau J M 1974 *Symp. Math.* **14** 343
[13] Kummer M 1982 *Commun. Math. Phys.* **84** 133
[14] Souriau J M 1970 *Structure des Systemes Dynamiques* (Paris: Dunod)
[15] Woodhouse N M J 1980 *Geometric Quantization* (Oxford: Oxford University Press)